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CYCLIC MONOPOLES, AFFINE TODA AND SPECTRAL CURVES

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Abstract. We show that any cyclically symmetric monopole is gauge equivalent to Nahm data given by Sutcliffe's ansatz, and so obtained from the affine Toda equations. Further the direction (the Ercolani-Sinha vector) and base point of the linearising flow in the Jacobian of the spectral curve associated to the Nahm equations arise as pull-backs of Toda data. A theorem of Accola and Fay then means that the theta-functions arising in the solution of the monopole problem reduce to the theta-functions of Toda.

Contents

1. Introduction

Magnetic monopoles, the topological soliton solutions of Yang-Mills-Higgs gauge theories in three space dimensions with particle-like properties, have been the subject of considerable interest over the years. BPS monopoles, arising from a limit in which the the Higgs potential is removed but a remnant of this remains in the boundary conditions, satisfy the first order Bogomolny equation

$$B_i = \frac{1}{2}\sum_{j,k=1}^{3} \epsilon_{ijk} F^{jk} = D_i \Phi$$

and have merited particular attention (see [MS04] for a recent review). This focus is in part due to the ubiquity of the Bogomolny equation. Here F_{ij} is the field strength associated to a gauge field A, and Φ is the Higgs field. We shall focus on the case when the gauge group is $SU(2)$. The Bogomolny equation may be viewed as a dimensional reduction of the four dimensional self-dual equations upon setting all functions independent of x_4 and identifying $\Phi = A_4$; they are also encountered in supersymmetric theories when requiring certain field configurations to preserve some fraction of supersymmetry. The study of BPS monopoles is intimately connected with integrable systems. Nahm gave a transform of the ADHM instanton construction to produce BPS monopoles [Nah82] and the resulting Nahm's equations have Lax form with corresponding spectral curve $\hat{\mathcal{C}}$. This curve, investigated by Corrigan and Goddard [CG81], was given a twistorial description by Hitchin [Hit82] where the same curve lies in mini-twistor space, $\hat{\mathcal{C}} \subset \mathrm{T}\mathbb{P}^1$. Just as Ward's twistor transform relates instanton solutions on $\mathbb{R}^4$ to certain holomorphic vector bundles over the twistor space $\mathbb{CP}^3$, Hitchin showed that the dimensional reduction leading to BPS monopoles could be made at the twistor level as well and was able to prove that all monopoles could be obtained by this approach [Hit83] provided the curve $\hat{\mathcal{C}}$ was subject to certain nonsingularity conditions. Bringing methods from integrable systems to bear upon the construction of solutions to Nahm's equations for the gauge group $SU(2)$ Ercolani and Sinha [ES89] later showed how one could solve (a gauge transform of) the Nahm equations in terms of a Baker-Akhiezer function for the curve $\hat{\mathcal{C}}$.

Although many general results have now been obtained few explicit solutions are known. This is for two reasons, each coming from a transcendental constraint on the curve $\hat{\mathcal{C}}$. The first is that the curve $\hat{\mathcal{C}}$ is subject to a set of constraints whereby the periods of a meromorphic differential on the curve are specified. This type of constraint arises in many other settings as well, for example when specifying the filling fractions of a curve in the AdS/CFT correspondence. Such constraints are transcendental in nature and until quite recently these had only been solved in the case of elliptic curves (which correspond to charge 2 monopoles). In [BE06, BE07] they were solved for a class of charge 3 monopoles using number theoretic results of Ramanujan. The second type of constraint is that the linear flow on the Jacobian of $\hat{\mathcal{C}}$ corresponding to the integrable motion only intersects the theta divisor in a prescribed manner. In the monopole setting this means the Nahm data will yield regular monopole solutions but a similar constraint also appears in other applications of integrable systems. In Hitchin's approach (reviewed below) this may be expressed as the vanishing of a real one parameter family of cohomologies of certain line bundles, $H^0(\hat{\mathcal{C}}, L^\lambda(n-2)) = 0$ for $\lambda \in (0, 2)$. Viewing the line bundles as points on the Jacobian this is equivalent to a real line segment not intersecting the theta divisor Θ of the curve. Indeed there are sections for $\lambda = 0, 2$ and the flow is periodic (mod 2) in λ and so we are interested in the number of times a real line intersects Θ. While techniques exist that count the number of intersections of a complex line with the theta divisor we are unaware of anything comparable in the real setting and again solutions have only been found for particular curves [BE09]. Thus the application of integrable systems techniques to the construction of monopoles and (indeed more generally) encounters two types of problem that each merit further study.

The present paper will use symmetry to reduce these problems to ones more manageable. Long ago monopoles of charge n with cyclic symmetry $\mathtt{C}_n$ were shown to exist [OR82] and more recently such monopoles were reconsidered [HMM95] from a variety of perspectives. The latter work indeed considered the case of monopoles with more general Platonic symmetries and for the case of tetrahedral, octahedral and icosahedral symmetry (where such monopoles exist) the curves were reduced to elliptic curves. (See [HS96a, HS96b, HS97] for development of this work.) Our first result is to strengthen work of Sutcliffe [Sut96]. Motivated by Seiberg-Witten theory Sutcliffe gave an ansatz for $\mathtt{C}_n$ symmetric monopoles in terms of $su(n)$ affine Toda theory. The spectral curve $\hat{\mathcal{C}}$ of a $\mathtt{C}_n$ symmetric monopole yields an n-fold unbranched cover of the hyperelliptic spectral curve $\mathcal{C}$ of the affine Toda theory, a spectral curve that arises in Seiberg-Witten theory describing the pure gauge $\mathcal{N} = 2$ supersymmetric $su(n)$ gauge theory. (We shall recall some properties of the Nahm construction and this relation between curves in section 2.) Sutcliffe's ansatz (section 3) shows how solutions to the affine Toda equations yield cyclically symmetric monopoles. Our first result proves that any cyclically symmetric monopole is gauge equivalent to Nahm data given by

Sutcliffe's ansatz, and so obtained from the affine Toda equations. We mention that Hitchin in an unpublished note had, prior to Sutcliffe, observed that cyclic charge 3 monopoles were equivalent to solutions of the affine Toda equations. The remainder of this paper shows that the relation between the Nahm data and the affine Toda system is much closer than simply that they yield the same equations of motion. The solution of an integrable system is typically expressed in terms of the straight line motion on the Jacobian of the system's spectral curve. Such a line is determined both by its direction and a point on the Jacobian. We shall show that both the direction (given by the Ercolani-Sinha vector, section 4) and point relevant for monopole solutions (section 5) are obtained as pull-backs of Toda data. This connection is remarkable and ties the geometry together in a very tight manner. Section 6 recalls a theorem of Accola and Fay that holds in precisely this setting, showing how the theta-functional solutions of the monopole reduce to precisely the theta-functional solutions of Toda. At this stage we have reduced the problem of constructing cyclically symmetric monopoles to one of determining hyperelliptic curves that satisfy the transcendental constraints described above. Though more manageable the problems are still formidable and a construction in the charge 3 setting will be described elsewhere [BDE]. We conclude with a discussion.

2. Monopoles

We shall briefly recall the salient features for constructing $su(2)$ monopoles of charge n. We begin with Nahm's construction [Nah82]. In generalizing the ADHM construction of instantons Nahm established an equivalence between nonsingular monopoles and what is now referred to as Nahm data: three $n\times n$ matrices $T_i(s)$ with $s\in[0,2]$ satisfying

N1 Nahm's equation

$$\frac{dT_i}{ds}=\frac{1}{2}\sum_{j,k=1}^{3}\epsilon_{ijk}[T_j,T_k], \tag{2.1}$$

N2 $T_i(s)$ is regular for $s\in(0,2)$ and has simple poles at $s=0$ and $s=2$, the residues of which form an irreducible n-dimensional representation of $su(2)$,

N3 $T_i(s)=-T_i^{\dagger}(s), \qquad T_i(s)=T_i^{t}(2-s)$.

Upon defining

$$\begin{aligned}A(\zeta)&=T_1+iT_2-2iT_3\zeta+(T_1-iT_2)\zeta^2\\ M(\zeta)&=-iT_3+(T_1-iT_2)\zeta\end{aligned}$$

we find that Nahm's equation is equivalent to the Lax equation

$$\frac{dT_i}{ds}=\frac{1}{2}\sum_{j,k=1}^{3}\epsilon_{ijk}[T_j,T_k]\Longleftrightarrow[\frac{d}{ds}+M,A]=0.$$

Here ζ is a spectral parameter. Following from the Lax equation we have the invariance of the spectral curve

$$\hat{\mathcal{C}}:\ 0=P(\eta,\zeta):=\det(\eta 1_n+A(\zeta)) \tag{2.2}$$

where

$$P(\eta,\zeta)=\eta^n+a_1(\zeta)\eta^{n-1}+\ldots+a_n(\zeta),\qquad \deg a_r(\zeta)\leq 2r. \tag{2.3}$$

As with any spectral curve presented in the form (2.2) one should always ask where $\hat{\mathcal{C}}$ lies. Typically the spectral curve lies in a surface, $\hat{\mathcal{C}}\subset\mathcal{S}$, and properties of the surface are closely allied with the integrable system encoded by the Lax equation. (For example, for suitable surfaces, the separation of variables may be described by $\mathrm{Hilb}^{[N]}(\mathcal{S})$, the Hilbert scheme of points on $\mathcal{S}$.) For the case at hand

$$\hat{\mathcal{C}} \subset T\mathbb{P}^1 := \mathcal{S}, \qquad (\eta,\zeta) \to \eta \frac{d}{d\zeta} \in T\mathbb{P}^1,$$

and monopoles admit a minitwistor description: the curve $\hat{\mathcal{C}}$ corresponds to those lines in $\mathbb{R}^3$ which admit normalizable solutions of an appropriate scattering problem in both directions [Hit82, Hit83]. This latter description makes clear that $\hat{\mathcal{C}}$ comes equipped with an antiholomorphic involution or real structure coming from the reversal of orientation of lines

$$(\eta,\zeta) \to (-\bar{\eta}/\bar{\zeta}^2, -1/\bar{\zeta}).$$

This means the coefficients of (2.3) are such that

$$a_r(\zeta) = (-1)^r \zeta^{2r} \overline{a_r(-1/\overline{\zeta}\,)} \tag{2.4}$$

and so each may be expressed in terms of $2r+1$ (real) parameters

$$a_r(\zeta) = \chi_r \left[\prod_{l=1}^{r} \left(\frac{\overline{\alpha}_{r,l}}{\alpha_{r,l}} \right)^{1/2} \right] \prod_{k=1}^{r} (\zeta - \alpha_{r,k})(\zeta + \frac{1}{\overline{\alpha}_{r,k}}), \qquad \alpha_{r,k} \in \mathbb{C}, \ \chi_r \in \mathbb{R}.$$

We remark that a real structure constrains the form of the period matrix of a curve and that while in general there may be between 0 and $\hat{g}+1$ ovals of fixed points of an antiholomorphic involution ($\hat{g}$ being the genus of $\hat{\mathcal{C}}$) for the case at hand there are no fixed points. For the monopole spectral curve (2.3) we have (generically) $\hat{g} = (n-1)^2$.

Although in many situations the solution of the integrable system encoded by a Lax pair (with spectral parameter) only depends on intrinsic properties of the spectral curve the monopole physical setting means that extrinsic properties of our curve in $T\mathbb{P}^1$ are relevant here. Spatial symmetries act on the monopole spectral curve via fractional linear transformations. Although a general Möbius transformation does not change the period matrix of a curve $\hat{\mathcal{C}}$ only the subgroup $PSU(2) < PSL(2,\mathbb{C})$ preserves the reality properties necessary for a monopole spectral curve. These reality conditions are an extrinsic feature of the curve (encoding the space-time aspect of the problem) whereas the intrinsic properties of the curve are invariant under birational transformations or the full Möbius group. Such extrinsic aspects are not a part of the usual integrable system story. Thus $SO(3)$ spatial rotations induce an action on $T\mathbb{P}^1$ via $PSU(2)$: if $\begin{pmatrix} p & q \\ -\bar{q} & \bar{p} \end{pmatrix} \in PSU(2)$, $(|p|^2 + |q|^2 = 1)$ then

$$\zeta \to \tilde{\zeta} := \frac{\bar{p}\,\zeta - \bar{q}}{q\,\zeta + p}, \qquad \eta \to \tilde{\eta} := \frac{\eta}{(q\,\zeta + p)^2} \tag{2.5}$$

corresponds to a rotation by θ around $\mathbf{n} \in S^2$ where

$$n_1 \sin(\theta/2) = \operatorname{Im} q, \ \ n_2 \sin(\theta/2) = -\operatorname{Re} q, \ \ n_3 \sin(\theta/2) = \operatorname{Im} p, \ \ \cos(\theta/2) = -\operatorname{Re} p.$$

This $SO(3)$ action commutes with the standard real structure on $T\mathbb{P}^1$. The action on the spectral curve may be expressed as

$$P(\tilde{\eta},\tilde{\zeta}) = \frac{\tilde{P}(\eta,\zeta)}{(q\,\zeta + p)^{2n}}, \qquad \tilde{P}(\eta,\zeta) = \eta^n + \sum_{r=1}^{n} \eta^{n-r}\,\tilde{a}_r(\zeta), \tag{2.6}$$

where in terms of the parameterization above

$$a_r(\zeta) \to \frac{\tilde{a}_r(\zeta)}{(q\,\zeta+p)^{2r}} \equiv \frac{\tilde{\chi}_r}{(q\,\zeta+p)^{2r}}\left[\prod_{l=1}^{r}\left(\frac{\bar{\tilde{\alpha}}_l}{\tilde{\alpha}_l}\right)^{1/2}\right]\prod_{k=1}^{r}(\zeta-\tilde{\alpha}_k)(\zeta+\frac{1}{\bar{\tilde{\alpha}}_k})$$

with

$$\alpha_k \to \tilde{\alpha}_k \equiv \frac{p\alpha_k+\bar{q}}{\bar{p}-\alpha_k q}, \qquad \chi_r \to \tilde{\chi}_r \equiv \chi_r \prod_{k=1}^{r}\left[\frac{(\bar{p}-\alpha_k q)(p-\bar{\alpha}_k\bar{q})(\bar{\alpha}_k\bar{p}+q)(\alpha_k p+\bar{q})}{\alpha_k\bar{\alpha}_k}\right]^{1/2}.$$

In particular the form of the curve does not change under a rotation: that is, if $a_r = 0$ then so also $\tilde{a}_r = 0$.

Hitchin, Manton and Murray [HMM95] showed how curves invariant under finite subgroups of $SO(3)$ or their binary covers yield symmetric monopoles. Suppose we have a symmetry; the spectral curve $0 = P(\eta,\zeta)$ is transformed to the same curve, $0 = P(\tilde{\eta},\tilde{\zeta}) = \tilde{P}(\eta,\zeta)/(q\,\zeta+p)^{2n}$. Then $P(\eta,\zeta) = \tilde{P}(\eta,\zeta)$, or equivalently $a_r(\zeta) = \tilde{a}_r(\zeta)$. Relevant for us is the example of cyclically symmetric monopoles. Let $\omega = \exp(2\pi i/n)$. A rotation of order n is then given by $\bar{p} = \omega^{1/2}$, $q = 0$ which yields

$$\phi:\ (\eta,\zeta) \to (\omega\eta,\omega\zeta).$$

Correspondingly $\eta^i\zeta^j$ is invariant for $i+j \equiv 0 \mod n$ and the spectral curve

$$\eta^n + a_1\eta^{n-1}\zeta + a_2\eta^{n-2}\zeta^2 + \ldots + a_n\zeta^n + \beta\zeta^{2n} + \gamma = 0$$

is invariant under the cyclic group $\mathtt{C}_n$ generated by this rotation. Imposing the reality conditions (2.4) and centering the monopole (setting $a_1 = 0$) then gives us the spectral curve in the form

$$\eta^n + a_2\eta^{n-2}\zeta^2 + \ldots + a_n\zeta^n + \beta\zeta^{2n} + (-1)^n\bar{\beta} = 0, \qquad a_i \in \mathbf{R} \tag{2.7}$$

and by an overall rotation we may choose β real.

Now the $\mathtt{C}_n$-invariant curve $\hat{\mathcal{C}}$ (2.7) of genus $\hat{g} = (n-1)^2$ is an n-fold unbranched cover of a genus $g = n-1$ curve $\mathcal{C}$. The Riemann-Hurwitz theorem yields the relation $\hat{g} = n(g-1)+1$. Introduce the rational invariants $x = \eta/\zeta$, $\nu = \zeta^n\beta$, then

$$x^n + a_2x^{n-2} + \ldots + a_n + \nu + \frac{(-1)^n|\beta|^2}{\nu} = 0$$

and upon setting $y = \nu - (-1)^n|\beta|^2/\nu$ we obtain the curve

$$y^2 = (x^n + a_2x^{n-2} + \ldots + a_n)^2 - 4(-1)^n|\beta|^2. \tag{2.8}$$

This curve is the spectral curve of $su(n)$ affine Toda theory in standard hyperelliptic form.

For future reference we note that the n-points $\hat{\infty}_j$ above the point $\zeta = \infty$ project to one of the infinite points, ∞_+, of the curve (2.8), while the n-points above the point $\zeta = 0$ project to the other infinite point. At $\hat{\infty}_j$ we have $\eta/\zeta \sim \rho_j\zeta$ as $\zeta \sim \hat{\infty}_j$, with $\rho_j = \beta^{1/n}\exp(2\pi i[j+1/2]/n)$.

The $n = 2$ example: The reality conditions for $n = 2$ and $a_2(\zeta) = \beta\zeta^4 + \gamma\zeta^2 + \delta$ means that $\delta = \bar{\beta}$ and $\gamma = \bar{\gamma}$ and (2.7) becomes

$$\eta^2 + \beta\zeta^4 + \gamma\zeta^2 + \bar{\beta} = 0.$$

This is an elliptic curve. If $\beta = |\beta|e^{2i\theta}$ let $U = \zeta e^{i\theta}$ and $V = i\eta e^{i\theta}/|\beta|^{1/2}$ and this may be rewritten as

$$V^2 = U^4 + t\,U^2 + 1, \qquad t = \gamma/|\beta|. \tag{2.9}$$

For irreducibility $t \neq 2$. Now the curve (2.8) becomes (with $Y = y/\sqrt{\gamma^2 - 4|\beta|^2}$)

$$Y^2 = x^4 + t'\,x^2 + 1, \qquad t' = \frac{2t}{\sqrt{t^2-4}}. \tag{2.10}$$

These two curves (2.9, 2.10) are 2-isogenous: if we quotient the former curve under the involution $(U,V) \to (-U,-V)$ we obtain the latter.

3. The Sutcliffe Ansatz

Some years ago Sutcliffe [Sut96] introduced the following ansatz for cyclically symmetric monopoles. Let

$$T_1 + iT_2 = \begin{pmatrix} 0 & e^{(q_1-q_2)/2} & 0 & \dots & 0 \\ 0 & 0 & e^{(q_2-q_3)/2} & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & e^{(q_{n-1}-q_n)/2} \\ e^{(q_n-q_1)/2} & 0 & 0 & \dots & 0 \end{pmatrix} \tag{3.1}$$

$$T_1 - iT_2 = -\begin{pmatrix} 0 & 0 & \dots & 0 & e^{(q_n-q_1)/2} \\ e^{(q_1-q_2)/2} & 0 & \dots & 0 & 0 \\ 0 & e^{(q_2-q_3)/2} & \dots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & e^{(q_{n-1}-q_n)/2} & 0 \end{pmatrix} \tag{3.2}$$

$$T_3 = -\frac{i}{2}\begin{pmatrix} p_1 & 0 & \dots & 0 \\ 0 & p_2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & p_n \end{pmatrix} \tag{3.3}$$

where p_i, q_i are real. Then $T_i(s) = -T_i^\dagger(s)$ and Nahm's equations yield

$$\frac{d}{ds}\left(T_1 + iT_2\right) = i[T_3, T_1 + iT_2] \qquad \Rightarrow \begin{cases} p_1 - p_2 = \dot q_1 - \dot q_2, \\ \qquad \vdots \\ p_n - p_1 = \dot q_n - \dot q_1. \end{cases}$$

$$\frac{d}{ds}T_3 = [T_1, T_2] = \frac{i}{2}[T_1 + iT_2, T_1 - iT_2] \qquad \Rightarrow \begin{cases} \dot p_1 = -e^{q_1-q_2} + e^{q_n-q_1}, \\ \qquad \vdots \\ \dot p_n = -e^{q_n-q_1} + e^{q_{n-1}-q_n}. \end{cases}$$

These equations then follow from the equations of motion of the affine Toda Hamiltonian

$$H = \frac{1}{2}\left(p_1^2 + \ldots + p_n^2\right) - \left[e^{q_1-q_2} + e^{q_2-q_3} + \ldots + e^{q_n-q_1}\right]. \tag{3.4}$$

Sutcliffe's observation is that particular solutions of these equations will then yield cyclically invariant monopoles. In fact the monopole Lax operator $A(\zeta)$ here is essentially the usual Toda Lax operator and

$$\frac{1}{2}\mathrm{Tr} A(\zeta)^2 = \zeta^2\, H.$$

The spectral curve of the affine Toda system is then (2.8) upon restricting the center of mass motion $\sum_i p_i = 0 = \sum_i q_i$. The constant β may be related to the coefficient of the scaling element when the Toda equations are expressed in terms of the affine algebra $\widehat{\mathrm{sl}}_n$.

In fact we may strengthen Sutcliffe's ansatz substantially. At this stage we only have that solutions of the Toda equations will yield some solutions of the Nahm equations with cyclic symmetry. First we will show that any $\mathtt{C}_n$ invariant solution of Nahm's equations (for charge n $su(2)$ monopoles) are given by solutions of the affine Toda equations. Then we will very concretely relate the solutions.

We have that $G \subset SO(3)$ acts on triples $\mathbf{t} = (T_1, T_2, T_3) \in \mathbb{R}^3 \otimes SL(n, \mathbb{C})$ via the natural action on $\mathbb{R}^3$ and conjugation on $SL(n, \mathbb{C})$. This natural action may be identified with the $SU(2)$ action on $\mathcal{O}(2)$ given above. If $g' \in SO(3)$ and $g = \rho(g')$ is its image in $SL(n, \mathbb{C})$ then we have

$$\begin{aligned} &g' \circ \left[\eta + (T_1 + iT_2) - 2iT_3\zeta + (T_1 - iT_2)\zeta^2\right] \\ &\quad = \omega\left[\eta + \omega^{-1} g(T_1 + iT_2)g^{-1} - 2igT_3g^{-1}\zeta + \omega g(T_1 - iT_2)g^{-1}\zeta^2\right]. \end{aligned}$$

Thus invariance of the spectral curve gives

$$\begin{aligned} g(T_1 + iT_2)g^{-1} &= \omega(T_1 + iT_2), \\ gT_3g^{-1} &= T_3, \\ g(T_1 - iT_2)g^{-1} &= \omega^{-1}(T_1 - iT_2). \end{aligned}$$

Now Hitchin, Manton and Murray [HMM95] have described how the $SO(3)$ action on $SL(n, \mathbb{C})$ decomposes as the direct sum $\underline{2n-1} \oplus \underline{2n-3} \oplus \ldots \oplus \underline{5} \oplus \underline{3}$ where $\underline{2k-1}$ denotes the $SO(3)$ irreducible representation of dimension $2k-1$. We may identify $SO(3)$ and its image in $SL(n, \mathbb{C})$ and because this decomposition has $\operatorname{rank} SL(n, \mathbb{C}) = n-1$ summands then, by a theorem of Kostant [K], the Lie algebra of this $SO(3)$ is a principal three-dimensional subalgebra. By conjugation we may express our generator g' of $\mathtt{C}_n$ as $g' = \exp\left[\dfrac{2\pi}{n}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right]$ and then $g = \rho(g') = \exp\left[\frac{2\pi}{n} H\right]$ where H is semi-simple and the generator of the principal three-dimensional algebra's Cartan subalgebra. Kostant described the action of such elements on arbitrary semi-simple Lie algebras and their roots. For the case at hand we have that g is equivalent to $\operatorname{Diag}(\omega^{n-1}, \ldots, \omega, 1)$ and that

$$gE_{ij}g^{-1} = \omega^{j-i}\, E_{ij}.$$

Therefore at this stage we know that for a cyclically invariant monopole we may write

$$T_1 + iT_2 = \sum_{\alpha \in \hat{\Delta}} e^{(\alpha, \tilde{q})/2}\, E_\alpha, \qquad T_3 = -\frac{i}{2}\sum_j \tilde{p}_j\, H_j$$

where in principle $\tilde{q}_i$, $\tilde{p}_i \in \mathbb{C}$, and $\alpha \in \hat{\Delta}$ are the simple roots together with minus the highest root. (The sum over H_i may be taken as either the Cartan subalgebra of $SL(n, \mathbb{C})$ or, by reinstating the center of mass, the Cartan subalgebra of $GL(n, \mathbb{C})$.) The Sutcliffe ansatz follows if the $\tilde{q}_i$ and $\tilde{p}_i$ may be chosen real. Now by an $SU(n)$ transformation $\operatorname{Diag}(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n})$ (where $\sum_i \theta_i = 0$) together with an overall $SO(3)$ rotation the reality of $\tilde{q}_i$ may be achieved. The reality of $\tilde{p}_i$ follows upon imposing $T_i(s) = -T_i^\dagger(s)$ which also fixes $T_1 - iT_2$. At this stage we have established the following.

Theorem 3.1. *Any cyclically symmetric monopole is gauge equivalent to Nahm data given by Sutcliffe's ansatz, and so obtained from the affine Toda equations.*

4. Flows and Solutions

The relation between the Nahm data and the affine Toda system is much closer than simply that they yield the same equations of motion. Let $\hat{\mathcal{C}}$ denote the genus $(n-1)^2$ spectral curve of the monopole and $\mathcal{C}$ denote the genus $n-1$ spectral curve of the Toda theory. We have already noted that $\mathcal{C} = \hat{\mathcal{C}}/\mathsf{C}_n$ and the natural projection $\pi : \hat{\mathcal{C}} \to \mathcal{C}$ is an n-fold unbranched cover. The solution of an integrable system is typically expressed in terms of the straight line motion on the Jacobian of the system's spectral curve. Such a line is determined both by its direction and a point on the Jacobian. We shall now show that both the direction and point relevant for monopole solutions are obtained as pull-backs of Toda data.

First we recall that meromorphic differentials describe flows, and that a meromorphic differential on a Riemann surface is uniquely specified by its singular parts together with some normalisation conditions. If $\left\{\hat{\mathfrak{a}}_i, \hat{\mathfrak{b}}_i\right\}_{i=1}^{\hat{g}}$ form a canonical basis for $H_1(\hat{\mathcal{C}}, \mathbb{Z})$,

$$\hat{\mathfrak{a}}_i \cap \hat{\mathfrak{b}}_j = -\hat{\mathfrak{b}}_j \cap \hat{\mathfrak{a}}_i = \delta_{ij},$$

then one such normalisation condition is that the $\hat{\mathfrak{a}}$-periods of the meromorphic differential vanish. (Thus the freedom to add to the meromorphic differential a holomorphic differential without changing its singular part is eliminated.) In what follows we denote by $\{\mathfrak{a}_i, \mathfrak{b}_i\}_{i=1}^{g}$ a similar canonical basis for $H_1(\mathcal{C}, \mathbb{Z})$.

For the monopole the Lax operator $A(\zeta)$ has poles at $\zeta = \infty$. If we denote $\hat{\infty}_j$ to be the n points on the spectral curve above $\zeta = \infty$ (and these may be assumed distinct) then we find that $\eta/\zeta = \rho_j \zeta$ as $\zeta \sim \hat{\infty}_j$. Consequently in terms of a local coordinate t at $\hat{\infty}_j$, $\zeta = 1/t$, then

$$d\left(\frac{\eta}{\zeta}\right) = \left(-\frac{\rho_j}{t^2} + O(1)\right) dt.$$

Thus on the monopole spectral curve we may uniquely define a meromorphic differential by the pole behaviour at $\hat{\infty}_j$ and normalization

$$\gamma_\infty = \left(\frac{\rho_j}{t^2} + O(1)\right) dt, \qquad 0 = \oint_{\hat{\mathfrak{a}}_i} \gamma_\infty.$$

The vector of $\hat{\mathfrak{b}}$-periods,

$$\widehat{\boldsymbol{U}} = \frac{1}{2i\pi} \oint_{\hat{\mathfrak{b}}} \gamma_\infty,$$

known as the Ercolani-Sinha vector [ES89], determines the direction of the monopole flow on $\mathrm{Jac}(\hat{\mathcal{C}})$. This vector is in fact constrained. Let us first recall Hitchin's conditions on a monopole spectral curve, equivalent to the Nahm data already given. These are

- **H1** Reality conditions $a_r(\zeta) = (-1)^r \zeta^{2r} \overline{a_r(-1/\overline{\zeta}\,)}$
- **H2** Let L^λ denote the holomorphic line bundle on $T\mathbb{P}^1$ defined by the transition function $g_{01} = \exp(-\lambda\eta/\zeta)$ and let $L^\lambda(m) \equiv L^\lambda \otimes \pi^*\mathcal{O}(m)$ be similarly defined in terms of the transition function $g_{01} = \zeta^m \exp(-\lambda\eta/\zeta)$. Then L^2 is trivial on $\hat{\mathcal{C}}$ and $L^1(n-1)$ is real.
- **H3** $H^0(\hat{\mathcal{C}}, L^\lambda(n-2)) = 0$ for $\lambda \in (0,2)$

We have already seen the reality conditions. Here the triviality of L^2 means that there exists a nowhere-vanishing holomorphic section. The following are equivalent [ES89, HMR00]:

(1) L^2 is trivial on $\hat{\mathcal{C}}$.

(2) $2\widehat{\boldsymbol{U}} \in \Lambda \Longleftrightarrow \widehat{\boldsymbol{U}} = \frac{1}{2\pi i}\left(\oint_{\hat{\mathfrak{b}}_1} \gamma_\infty, \ldots, \oint_{\hat{\mathfrak{b}}_{\hat{g}}} \gamma_\infty\right)^T = \frac{1}{2}\boldsymbol{n} + \frac{1}{2}\hat{\tau}\boldsymbol{m}.$

(3) There exists a 1-cycle $\widehat{\mathfrak{es}} = \boldsymbol{n}\cdot\hat{\mathfrak{a}} + \boldsymbol{m}\cdot\hat{\mathfrak{b}}$ such that for every holomorphic differential

$$\Omega = \frac{\beta_0\eta^{n-2} + \beta_1(\zeta)\eta^{n-3} + \ldots + \beta_{n-2}(\zeta)}{\partial\mathcal{P}/\partial\eta}\, d\zeta, \qquad \oint_{\widehat{\mathfrak{es}}} \Omega = -2\beta_0.$$

Here $\hat{\tau}$ is the period matrix of $\hat{\mathcal{C}}$ and Λ is the associated period lattice of the curve. Thus $\widehat{\boldsymbol{U}}$ is constrained to be a half-period. These are known as the Ercolani-Sinha constraints and they impose $\hat{g}$ *transcendental constraints* on the curve yielding

$$\sum_{j=2}^{n}(2j+1) - \hat{g} = (n+3)(n-1) - (n-1)^2 = 4(n-1)$$

degrees of freedom.

We now turn to consider the behaviour of the Ercolani-Sinha vector under a symmetry. Clearly our group acting the curve leads to an action on divisors and consequently on the Jacobian. We now show that the Ercolani-Sinha vector describing the flow is fixed under the symmetry. This means the vector may be obtained from the pull-back of a vector on the Jacobian of the quotient (Toda) curve.

Suppose we have a symmetry

$$0 = P(\eta,\zeta) = P(\tilde{\eta},\tilde{\zeta}) = \frac{\tilde{P}(\eta,\zeta)}{(q\,\zeta+p)^{2n}}.$$

In particular

$$\partial_{\tilde{\eta}}P(\tilde{\eta},\tilde{\zeta}) = (q\,\zeta+p)^2\partial_\eta P(\tilde{\eta},\tilde{\zeta}) = \frac{\partial_\eta\tilde{P}(\eta,\zeta)}{(q\,\zeta+p)^{2n-2}} = \frac{\partial_\eta P(\eta,\zeta)}{(q\,\zeta+p)^{2n-2}}. \tag{4.1}$$

Using

$$d\tilde{\zeta} = \frac{d\zeta}{(q\,\zeta+p)^2}$$

we see then that

$$\frac{\tilde{\zeta}^r\tilde{\eta}^s d\tilde{\zeta}}{\partial_{\tilde{\eta}}P(\tilde{\eta},\tilde{\zeta})} = \frac{(\bar{p}\zeta-\bar{q})^r(q\,\zeta+p)^{2n-4-r-2s}\eta^s d\zeta}{\partial_\eta P(\eta,\zeta)}.$$

Bringing these together

Lemma 4.1. *The differential* $\hat{\omega}_{r,s} = \dfrac{\zeta^r\eta^s d\zeta}{\partial_\eta P(\eta,\zeta)}$ *is invariant under the rotation (2.5) if and only if*

$$\zeta^r = (\bar{p}\zeta-\bar{q})^r(q\,\zeta+p)^{2n-4-r-2s}.$$

This always has a solution, the holomorphic differential

$$\hat{\omega} = \frac{\eta^{n-2}d\zeta}{\partial_\eta P(\eta,\zeta)}.$$

For the particular case of interest here, for rotations given by $q=0$, $|p|^2=1$, then

$$\phi^*\left(\frac{\zeta^r\eta^s d\zeta}{\partial_\eta P(\eta,\zeta)}\right) = \omega^{r+s+2}\,\frac{\zeta^r\eta^s d\zeta}{\partial_\eta P(\eta,\zeta)} \tag{4.2}$$

and we also have solutions for each s ($0\le s\le n-2$) and $r=n-2-s$. These give us $g=n-1$ $\mathtt{C}_n$-invariant holomorphic differentials which are pullbacks of the holomorphic differentials on $\mathcal{C}$. We remark also that the symmetry always fixes the subspaces $\sum_r \mu_r\omega_{r,s}$ for fixed s.

Thus on the space of holomorphic differentials $\{\hat\omega_I\}_{I=1}^{\hat g-1}\cup\{\hat\omega_{0,n-2}\}$ (for appropriate $I=(r,s)$ whose order does not matter) we have
(4.3)
$$\phi^*(\hat\omega_1,\ldots,\hat\omega_{\hat g-1},\hat\omega_{0,n-2})=(\hat\omega_1,\ldots,\hat\omega_{\hat g-1},\hat\omega_{0,n-2})\begin{pmatrix}* & * & 0\\ * & * & 0\\ 0 & 0 & 1\end{pmatrix}:=(\hat\omega_1,\ldots,\hat\omega_{\hat g-1},\hat\omega_{0,n-2})L$$
where L is a $\hat g\times\hat g$ complex matrix. As $L^n=1$, the matrix is both invertible and diagonalizable.

With $\{\hat{\mathfrak a}_i,\hat{\mathfrak b}_i\}$ the canonical homology basis introduced earlier and $\{\hat u_j\}$ a basis of holomorphic differentials for our Riemann surface $\hat{\mathcal C}$ we have the matrix of periods

$$\begin{pmatrix}\oint_{\hat{\mathfrak a}_i}\hat u_j\\ \oint_{\hat{\mathfrak b}_i}\hat u_j\end{pmatrix}=\begin{pmatrix}\hat{\mathcal A}\\ \hat{\mathcal B}\end{pmatrix}=\begin{pmatrix}1\\ \hat\tau\end{pmatrix}\hat{\mathcal A} \tag{4.4}$$

with $\hat\tau=\hat{\mathcal B}\hat{\mathcal A}^{-1}$ the period matrix. If σ is any automorphism of $\hat{\mathcal C}$ then σ acts on $H_1(\hat{\mathcal C},\mathbb Z)$ and the holomorphic differentials by

$$\sigma_*\begin{pmatrix}\hat{\mathfrak a}_i\\ \hat{\mathfrak b}_i\end{pmatrix}=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}\hat{\mathfrak a}_i\\ \hat{\mathfrak b}_i\end{pmatrix},\qquad \sigma^*\hat u_j=\hat u_k L^k_j,$$

where $\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in Sp(2\hat g,\mathbb Z)$ and $L\in GL(\hat g,\mathbb C)$. Then from

$$\oint_{\sigma_*\gamma}\hat u=\oint_\gamma\sigma^*\hat u$$

we obtain

$$\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}\hat{\mathcal A}\\ \hat{\mathcal B}\end{pmatrix}=\begin{pmatrix}\hat{\mathcal A}\\ \hat{\mathcal B}\end{pmatrix}L. \tag{4.5}$$

With the ordering of holomorphic differentials of (4.3) the second of the equivalent conditions for the Ercolani-Sinha vector says there exist integral vectors $\mathbf n$, $\mathbf m$ such that

$$(\mathbf n,\mathbf m)\begin{pmatrix}\hat{\mathcal A}\\ \hat{\mathcal B}\end{pmatrix}=-2(0,\ldots,0,1). \tag{4.6}$$

Now suppose σ corresponds to a symmetry coming from a rotation. Then the form of L in (4.3) gives

$$(\mathbf n,\mathbf m)\begin{pmatrix}\hat{\mathcal A}\\ \hat{\mathcal B}\end{pmatrix}=-2(0,\ldots,0,1)=-2(0,\ldots,0,1).L=(\mathbf n,\mathbf m)\begin{pmatrix}\hat{\mathcal A}\\ \hat{\mathcal B}\end{pmatrix}.L=(\mathbf n,\mathbf m)\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}\hat{\mathcal A}\\ \hat{\mathcal B}\end{pmatrix}$$

and so

$$\left((\mathbf n,\mathbf m)-(\mathbf n,\mathbf m)\begin{pmatrix}A & B\\ C & D\end{pmatrix}\right)\begin{pmatrix}\hat{\mathcal A}\\ \hat{\mathcal B}\end{pmatrix}=0.$$

As the rows of the lattice generated by $\begin{pmatrix}\hat{\mathcal A}\\ \hat{\mathcal B}\end{pmatrix}$ are independent over $\mathbb Z$ we therefore have that

$$(\mathbf n,\mathbf m)=(\mathbf n,\mathbf m)\begin{pmatrix}A & B\\ C & D\end{pmatrix}$$

for all symplectic matrices $\begin{pmatrix}A & B\\ C & D\end{pmatrix}$ representing the symmetries coming from spatial rotations. In particular $(\mathbf n,\mathbf m)$ is invariant under the group of symmetries. Therefore the Ercolani-Sinha vector is invariant and so as an element of the Jacobian, this will reduce to a vector of the Jacobian of the quotient curve. Viewing this vector as a divisor on the curve it projects to a divisor on the quotient curve. Thus we have established

Theorem 4.2. *The Ercolani-Sinha vector is invariant under the group of symmetries of the spectral curve arising from rotations (2.5),*

$$\widehat{\boldsymbol{U}} = \pi^*(\boldsymbol{U}), \qquad \boldsymbol{U} \in \mathrm{Jac}(\mathcal{C}). \tag{4.7}$$

For the cyclic symmetry under consideration we have from

$$dy = n\left(\nu + \frac{(-1)^n|\beta|^2}{\nu}\right)\frac{d\zeta}{\zeta} = -n(x^n + a_2x^{n-2} + \ldots + a_n)\frac{d\zeta}{\zeta},$$
$$\partial_\eta P(\eta,\zeta) = \zeta^{n-1}\partial_x(x^n + a_2x^{n-2} + \ldots + a_n),$$

that

$$\frac{\zeta^{n-2-s}\eta^s d\zeta}{\partial_\eta P(\eta,\zeta)} = \pi^*\left(-\frac{1}{n}\,\frac{x^s dx}{y}\right). \tag{4.8}$$

Thus each of the invariant differentials (for $0 \leq s \leq n-2$) reduce to hyperelliptic differentials.

5. The base point

In the construction of monopoles there is a distinguished point $\widetilde{\boldsymbol{K}} \in \mathrm{Jac}(\hat{\mathcal{C}})$ that Hitchin uses to identify degree $\hat{g}-1$ line bundles with $\mathrm{Jac}(\hat{\mathcal{C}})$. For $n \geq 3$ this point is a singular point of the theta divisor, $\widetilde{\boldsymbol{K}} \in \Theta_{\text{singular}}$ [BE06]. If we denote the Abel map by

$$\mathcal{A}_{\hat{Q}}(\hat{P}) = \int_{\hat{Q}}^{\hat{P}} \hat{u}_i$$

then

$$\widetilde{\boldsymbol{K}} = \hat{\boldsymbol{K}}_{\hat{Q}} + \mathcal{A}_{\hat{Q}}\left((n-2)\sum_{k=1}^{n} \hat{\infty}_k\right). \tag{5.1}$$

Here $\hat{\boldsymbol{K}}_{\hat{Q}}$ is the vector of Riemann constants for the curve $\hat{\mathcal{C}}$. If $\mathcal{K}_{\hat{\mathcal{C}}}$ is the canonical divisor of the curve then $\mathcal{A}_{\hat{Q}}(\mathcal{K}_{\hat{\mathcal{C}}}) = -2\hat{\boldsymbol{K}}_{\hat{Q}}$. The righthand side of (5.1) is in fact independent of the base point $\hat{Q}$ in its definition.

The point $\widetilde{\boldsymbol{K}}$ is the base point of the linear motion in the Jacobian referred to earlier and we shall now relate this to a point in the Jacobian of the Toda spectral curve $\mathcal{C}$. Let $\mathcal{A}_Q(\mathcal{K}_{\mathcal{C}}) = -2\boldsymbol{K}_Q$ be the corresponding quantities for the curve $\mathcal{C}$ with basis of holomorphic differentials $\{u_a\}$. We first relate $\pi^*\boldsymbol{K}_Q$ and $\hat{\boldsymbol{K}}_{\hat{Q}}$ where $\pi(\hat{Q}) = Q$ is some preimage of Q. Let our symmetry be $\phi : \hat{\mathcal{C}} \to \hat{\mathcal{C}}$, $\phi^n = 1$, and observe that (with $\pi(\hat{P}) = P$, $\pi(\hat{Q}) = Q$)

$$\pi^*(\mathcal{A}_Q(P)) = \pi^*\left(\int_Q^P u\right) = \sum_{s=0}^{n-1}\int_{\phi^s(\hat{Q})}^{\phi^s(\hat{P})} \hat{u} = \sum_{s=0}^{n-1}\left[\mathcal{A}_{\hat{Q}}\left(\phi^s(\hat{P})\right) - \mathcal{A}_{\hat{Q}}\left(\phi^s(\hat{Q})\right)\right].$$

(This is actually independent of the base-point chosen for the Abel map, so well-defined.) Now if $\sum_{\alpha=1}^{2g-2} P_\alpha$ is a canonical divisor for $\mathcal{C}$ then $\sum_{\alpha=1}^{2g-2}\sum_{s=0}^{n-1} \phi^s(\hat{P}_\alpha)$ is a canonical divisor for $\hat{\mathcal{C}}$. Thus

$$\begin{aligned}\pi^*(-2\boldsymbol{K}_Q) &= \pi^*\left(\mathcal{A}_Q(\mathcal{K}_{\mathcal{C}})\right)\\ &= \pi^*\left(\sum_{\alpha=1}^{2g-2}\int_Q^{P_\alpha} u\right)\end{aligned}$$

$$= \mathcal{A}_{\hat{Q}}(\hat{\mathcal{K}}_{\hat{\mathcal{C}}}) - 2(g-1)\sum_{s=0}^{n-1} \mathcal{A}_{\hat{Q}}\left(\phi^s(\hat{Q})\right)$$
$$= -2\hat{\boldsymbol{K}}_{\hat{Q}} - 2(g-1)\sum_{s=0}^{n-1} \mathcal{A}_{\hat{Q}}\left(\phi^s(\hat{Q})\right).$$

Therefore

$$\pi^*(\boldsymbol{K}_Q) = \hat{\boldsymbol{K}}_{\hat{Q}} + (g-1)\sum_{s=0}^{n-1} \mathcal{A}_{\hat{Q}}\left(\phi^s(\hat{Q})\right) + \hat{e}, \tag{5.2}$$

where $2\hat{e} \in \Lambda$ is a half-period. This expression may be rewritten as

$$\begin{aligned}
\pi^*(\boldsymbol{K}_Q) &= \hat{\boldsymbol{K}}_{\hat{Q}} + (g-1)\sum_{s=0}^{n-1} \mathcal{A}_{\hat{Q}}\left(\phi^s(\hat{Q})\right) + \hat{e} \\
&= \left[\hat{\boldsymbol{K}}_{\hat{Q}} + (\hat{g}-1)\mathcal{A}_{\hat{Q}}(\hat{P})\right] - (\hat{g}-1)\mathcal{A}_{\hat{Q}}(\hat{P}) + (g-1)\sum_{s=0}^{n-1} \mathcal{A}_{\hat{Q}}\left(\phi^s(\hat{Q})\right) + \hat{e} \\
&= \hat{\boldsymbol{K}}_{\hat{P}} - n(g-1)\mathcal{A}_{\hat{Q}}(\hat{P}) + (g-1)\sum_{s=0}^{n-1} \mathcal{A}_{\hat{Q}}\left(\phi^s(\hat{Q})\right) + \hat{e} \\
&= \hat{\boldsymbol{K}}_{\hat{P}} + (g-1)\sum_{s=0}^{n-1} \mathcal{A}_{\hat{P}}\left(\phi^s(\hat{Q})\right) + \hat{e},
\end{aligned}$$

showing the left-hand side is independent of the choice of base-point for the Abel map.

Comparison of (5.1) and (5.2) now shows that

$$\widetilde{\boldsymbol{K}} = \pi^*(\boldsymbol{K}_{\infty_+}) - \hat{e} \tag{5.3}$$

where $\pi(\hat{\infty}_k) = \infty_+$ as noted earlier. Now the half-period $\hat{e}$ can be identified and is of the form $\hat{e} = \pi^*(e)$. The actual identification depends on an homology choice and will be given in the next section, but for the moment we simply note the form

$$\widetilde{\boldsymbol{K}} = \pi^*(\boldsymbol{K}_{\infty_+} - e). \tag{5.4}$$

6. Fay-Accola factorization

The standard reconstruction of solutions for an integrable system with spectral curve $\hat{\mathcal{C}}$ proceeds by constructing the Baker-Akhiezer functions for this curve. These may be calculated in terms of theta functions for the curve and for our present purposes we may focus on the theta function $\theta(\lambda\widehat{\boldsymbol{U}} - \widetilde{\boldsymbol{K}}\,|\,\hat{\tau})$. This describes a flow on the Jacobian of $\hat{\mathcal{C}}$ in the direction of the Ercolani-Sinha vector $\widehat{\boldsymbol{U}}$ with base point $\widetilde{\boldsymbol{K}}$. We have observed that we have a cyclic unramified covering $\pi : \hat{\mathcal{C}} \to \mathcal{C}$ of the affine Toda spectral curve by the monopole spectral curve. The map π leads to a map $\pi^* : \mathrm{Jac}(\mathcal{C}) \to \mathrm{Jac}(\hat{\mathcal{C}})$ which may be lifted to $\pi^* : \mathbb{C}^g \to \mathbb{C}^{\hat{g}}$. Further we have established that

$$\lambda\widehat{\boldsymbol{U}} - \widetilde{\boldsymbol{K}} = \pi^*(\lambda\boldsymbol{U} - \boldsymbol{K}_{\infty_+} + e).$$

We now are in a position to make use of a remarkable factorization theorem due to Accola and Fay [Acc71, Fay73] and also observed by Mumford. When $\hat{z} = \pi^* z$ the theta functions on $\hat{\mathcal{C}}$ and $\mathcal{C}$ are related by this factorization theorem,

Theorem 6.1 (**Fay-Accola).** *With respect to the ordered canonical homology bases* $\{\hat{\mathfrak{a}}_i^c, \hat{\mathfrak{b}}_i^c\}$ *described below and for arbitrary* $\boldsymbol{z} = \in \mathbb{C}^g$ *we have*

$$\frac{\theta[\hat{e}](\pi^* \boldsymbol{z}; \hat{\tau}^c)}{\prod_{k=0}^{n-1} \theta \begin{bmatrix} 0 & 0 & \dots & 0 \\ \frac{k}{n} & 0 & \dots & 0 \end{bmatrix} (\boldsymbol{z}; \tau^c)} = c_0(\widehat{\tau}^c) \tag{6.1}$$

is a non-zero modular constant $c_0(\hat{\tau}^c)$ *independent of* $\boldsymbol{z}$. *Here* $\hat{\tau}^c$ *is the* $\mathfrak{a}$*-normalized period matrix for the curve* $\hat{\mathcal{C}}$ *in this homology basis and*

$$\hat{e} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ \frac{n-1}{2} & 0 & \dots & 0 \end{bmatrix} = \pi^* (e) = \pi^* \left(\begin{bmatrix} 0 & 0 & \dots & 0 \\ \frac{n-1}{2n} & 0 & \dots & 0 \end{bmatrix} \right).$$

The significance of this theorem for our setting is that it means we can reduce the construction of solutions to that of quantities purely in terms of the hyperelliptic affine Toda spectral curve.

The theorem is expressed in terms of a particular choice of homology basis which is well adapted to the symmetry at hand. In terms of the conformal automorphism $\phi : \hat{\mathcal{C}} \to \hat{\mathcal{C}}$ of $\hat{\mathcal{C}}$ that generates the group $\mathtt{C}_n = \{\phi^s \,|\, 0 \le s \le n-1\}$ of cover transformations of $\hat{\mathcal{C}}$ and the projection $\pi : \hat{\mathcal{C}} \to \mathcal{C}$ there exists a basis $\{\hat{\mathfrak{a}}_0^c, \hat{\mathfrak{b}}_0^c, \hat{\mathfrak{a}}_1^c, \hat{\mathfrak{b}}_1^c, \dots, \hat{\mathfrak{a}}_{\hat{g}-1}^c, \hat{\mathfrak{b}}_{\hat{g}-1}^c\}$ of homology cycles for $\hat{\mathcal{C}}$ and $\{\mathfrak{a}_0^c, \mathfrak{b}_0^c, \mathfrak{a}_1^c, \mathfrak{b}_1^c, \dots, \mathfrak{a}_{g-1}^c, \mathfrak{b}_{g-1}^c\}$ for $\mathcal{C}$ such that (for $1 \le j \le g-1,\ 0 \le s \le n$)

$$\begin{aligned} &\pi(\hat{\mathfrak{a}}_0^c) = \mathfrak{a}_0^c, \quad \pi(\hat{\mathfrak{a}}_{j+s(g-1)}^c) = \mathfrak{a}_j^c, && \pi(\hat{\mathfrak{b}}_0^c) = n\, \mathfrak{b}_0^c, \quad \pi(\hat{\mathfrak{b}}_{j+s(g-1)}^c) = \mathfrak{b}_j^c, \\ &\phi^s(\hat{\mathfrak{a}}_0^c) \sim \hat{\mathfrak{a}}_0^c, \qquad \phi^s(\hat{\mathfrak{a}}_j^c) = \hat{\mathfrak{a}}_{j+s(g-1)}^c, && \phi^s(\hat{\mathfrak{b}}_0) = \hat{\mathfrak{b}}_0^c, \qquad \phi^s(\hat{\mathfrak{b}}_j^c) = \hat{\mathfrak{b}}_{j+s(g-1)}^c. \end{aligned}$$

Here $\phi^s(\hat{\mathfrak{a}}_0)$ is homologous to $\hat{\mathfrak{a}}_0$. If $\hat{v}_i$ are the $\hat{\mathfrak{a}}$-normalized differentials for $\hat{\mathcal{C}}$, then

$$\delta_{i,j+s(g-1)} = \int_{\hat{\mathfrak{a}}_{j+s(g-1)}} \hat{v}_i = \int_{\phi^s(\hat{\mathfrak{a}}_j)} \hat{v}_i = \int_{\hat{\mathfrak{a}}_j} (\phi^s)^* \hat{v}_i = \int_{\hat{\mathfrak{a}}_j} \hat{v}_{i-s(g-1)},$$

and we find that

$$(\phi^s)^* \hat{v}_0 = \hat{v}_0, \qquad (\phi^s)^* \hat{v}_i = \hat{v}_{i-s(g-1)}. \tag{6.2}$$

If v_i are the normalized differentials for $\mathcal{C}$, then

$$\delta_{ij} = \int_{\mathfrak{a}_j} v_i = \int_{\pi(\hat{\mathfrak{a}}_{j+s(g-1)})} v_i = \int_{\mathfrak{a}_{j+s(g-1)}} \pi^*(v_i)$$

shows that

$$\pi^*(v_i) = \hat{v}_i + (\phi)^* \hat{v}_i + \dots + (\phi^{p-1})^* \hat{v}_i$$

and similarly that

$$\pi^*(v_0) = \hat{v}_0.$$

We may use the characters of $\mathtt{C}_n$ to construct the remaining linearly independent differentials on $\hat{\mathcal{C}}$.

From (6.2) we have an action of $\mathtt{C}_n$ on $\mathrm{Jac}(\hat{\mathcal{C}})$ which lifts to an automorphism of $\mathbb{C}^{\hat{g}}$ by

$$\phi^s(\hat{z}) = (\hat{z}_0, \hat{z}_{1-s(g-1)}, \dots, \hat{z}_{g-1-s(g-1)}, \dots, \hat{z}_{1+(p-s-1)(g-1)}, \dots, \hat{z}_{g-1+(p-s-1)(g-1)}) \tag{6.3}$$

Now (6.3) together with the invariance of the Ercolani-Sinha vector mans that in this cyclic homology basis we have

$$(\mathbf{n}, \mathbf{m}) = (r_0, \mathbf{r}, \dots, \mathbf{r}, s_0, \mathbf{s}, \dots, \mathbf{s}) \tag{6.4}$$

where the vectors $\mathbf{r} = (r_1, \dots, r_{g-1})$ and similarly $\mathbf{s}$ are each repeated n times. We also have

$$\pi_*(\widehat{\mathfrak{es}}) = r_0 \mathfrak{a}_0 + n\mathbf{r} \cdot \boldsymbol{\mathfrak{a}} + n s_0 \mathfrak{b}_0 + n\mathbf{s} \cdot \boldsymbol{\mathfrak{b}}. \tag{6.5}$$

With the choices above (things are different for $\hat{\mathfrak{b}}$-normalization) we may lift the map $\pi^* : \mathrm{Jac}(\mathcal{C}) \to \mathrm{Jac}(\hat{\mathcal{C}})$ to $\pi^* : \mathbb{C}^g \to \mathbb{C}^{\hat{g}}$,

$$\pi^*(z) = \pi^*(z_0, z_1, \ldots, z_{g-1}) = (n\, z_0, z_1, \ldots, z_{g-1}, \ldots, z_1, \ldots, z_{g-1}) = \hat{z}.$$

With this homology basis the period matrices for the two curves are related by the block form

$$\hat{\tau}^c = \begin{pmatrix} n\,\tau^c_{00} & \tau^c_{0j} & \tau^c_{0j} & \cdots & \tau^c_{0j} \\ \tau^c_{j0} & \mathcal{M} & \mathcal{M}^{(1)} & & \mathcal{M}^{(n-1)} \\ \vdots & & & & \\ \tau^c_{j0} & \mathcal{M}^{(1)} & & & \mathcal{M} \end{pmatrix}$$

where $\mathcal{M}^{(s)} = \int_{\phi^{-s}(\hat{\mathfrak{b}}_j)} \hat{v}_i$. The (r,s) block here has entry $\mathcal{M}^{s-r}$ and $(\mathcal{M}^{(s-r)})^T = \mathcal{M}^{(r-s)}$ by the bilinear identity. Then $\tau^c_{ij} = \sum_{s=0}^{n-1} \mathcal{M}^{(s)}_{ij}$. The case $n = 3$ is instructive, for here the $n-2$ block matrices are just numbers and we have

$$\hat{\tau}^c = \begin{pmatrix} a & b & b & b \\ b & c & d & d \\ b & d & c & d \\ b & d & d & c \end{pmatrix}, \qquad \tau^c = \begin{pmatrix} \frac{1}{3}a & b \\ b & c+2d \end{pmatrix}. \tag{6.6}$$

The point to note is that although the period matrix for $\hat{\mathcal{C}}$ involves integrations of differentials that do not reduce to hyperelliptic integrals, the combination of terms appearing in the reduction can be expressed in terms of hyperelliptic integrals. This is a definite simplification. Further the Θ function defined by $\hat{\tau}^c$ has the symmetries

$$\Theta(\hat{z}|\hat{\tau}^c) = \Theta(\phi^s(\hat{z})|\hat{\tau}^c)$$

for all $\hat{z} \in \mathbb{C}^{\hat{g}}$. In particular, the Θ divisor is fixed under $\mathtt{C}_n$.

If we are to reduce the construction of cyclic monopoles to a problem involving only hyperelliptic quantities we must describe the Ercolani-Sinha constaints in the context of the curve $\mathcal{C}$.

Theorem 6.2. *The Ercolani-Sinha constraint on the curve $\hat{\mathcal{C}}$ yields the constraint*

$$-2(0, \ldots, 0, 1) = (r_0, n\mathbf{r}, ns_0, n\mathbf{s}) \begin{pmatrix} \mathcal{A} \\ \mathcal{B} \end{pmatrix} \tag{6.7}$$

on the curve $\mathcal{C}$ with respect to the differentials $u_s = -x^s dx/(ny)$ $(s = 0, \ldots, n-2)$.

Proof. The invariance of the Ercolani-Sinha vector means that $\phi^*(\widehat{\mathfrak{es}}) = \widehat{\mathfrak{es}}$. Thus

$$\int_{\widehat{\mathfrak{es}}} \hat{\omega}_{r,s} = \int_{\phi^*(\widehat{\mathfrak{es}})} \hat{\omega}_{r,s} = \int_{\widehat{\mathfrak{es}}} \phi^* \hat{\omega}_{r,s} = \omega^{r+s+2} \int_{\widehat{\mathfrak{es}}} \hat{\omega}_{r,s},$$

where we have used (4.2). Thus the integral of any noninvariant differential around the cycle $\widehat{\mathfrak{es}}$ must vanish, while from (4.8) and the Ercolani-Sinha condition we have that

$$-2\,\delta_{s,n-2} = \int_{\widehat{\mathfrak{es}}} \pi^* \left(-\frac{1}{n} \frac{x^s dx}{y} \right) = \int_{\pi_*(\widehat{\mathfrak{es}})} -\frac{1}{n} \frac{x^s dx}{y}.$$

The theorem then follows upon using (6.5). □

In actual calculations it is convenient to use the unnormalized differentials $\hat{\omega}_{r,s}$ and $x^s dx/(ny)$ rather than Fay's normalized differentials $\hat{v}_i$. An alternate proof of Theorem 6.2 via Poincaré's reducibility theorem is given in the Appendix, which provides further useful relations amongst the periods of the two curves.

7. Discussion

In this paper we have shown that any cyclically symmetric monopole is gauge equivalent to Nahm data obtained via Sutcliffe's ansatz from the affine Toda equations. Further, the data needed to reconstruct the monopole, the Ercolani-Sinha vector and base point for linear flow on the Jacobian, may also be obtained from data on the affine Toda equation's hyperelliptic spectral curve $\mathcal{C}$. A theorem of Fay and Accola then enables us to express the theta functions for the monopole spectral curve in terms of the theta functions for the curve $\mathcal{C}$. Finally the transcendental constraints on the monopole's spectral curve can be recast as transcendental constraints for the hyperelliptic curve $\mathcal{C}$ (Theorem 6.2). At this stage then the construction of cyclically symmetric monopoles has been reduced to one entirely in terms of hyperelliptic curves. Although analogues of both the transcendental constraints still exist this is a significant simplification. We note that the structure of the theta divisor is better understood in the hyperelliptic setting [V95] and the hyperelliptic integrals are somewhat simpler than the general integrals appearing in the Ercolani-Sinha constraint for the full monopole curve.

Other approaches to constructing monopoles are known. In particular [HMM95] describe cyclically symmetric monopoles within the rational map approach (see also [MS04, §8.8]). These authors show that the rational map for monopoles with $\mathtt{C}_n$ invariance about the x_3-axis takes the form

$$R(z) = \frac{\mu z^l}{z^n - \nu}$$

where $0 \leq l \leq n-1$. The complex quantity ν determines μ when the monopoles are strongly centred. Here $\nu = (-1)^{n-1}\bar{\beta}$ of equation (2.7). The moduli space $\mathcal{M}_n^l$ is a 4-dimensional totally geodesic submanifold of the full moduli space. It is interesting that both the rational map description and the description we have presented lead to extra discrete parameters (l in the case of rational maps, and k in 6.1). The connection, if any, between these will be pursued elsewhere [BDE].

Clearly the ansatz for monopoles extends to other algebras. If we construct the spectral curve from the D_n Toda system using the $2n$ dimensional representation we find a spectral curve $\hat{\mathcal{C}}$ of the form

$$\eta^{2n} + a_1\eta^{2n-2}\zeta^2 + a_2\eta^{2n-4}\zeta^4 + \ldots + a_n\zeta^{2n} + \alpha\eta^2\left(\frac{1}{w} + \zeta^{4n-4}w\right) = 0.$$

Letting $x = \eta/\zeta$ the curve (upon dividing by ζ^{2n}) becomes

$$x^{2n} + a_1x^{2n-2} + a_2x^{2n-4} + \ldots + a_n + \alpha x^2\left(\frac{1}{w\zeta^{2n-2}} + \zeta^{2n-2}w\right) = 0.$$

and so we get with $\nu = \alpha w\zeta^{2n-2}$

$$P_n(x^2) + x^2\left(\nu + \frac{\alpha^2}{\nu}\right) = 0$$

leading to a hyperelliptic curve $\tilde{\mathcal{C}}$

$$y^2 = P_n(x^2)^2 - 4\alpha^2x^4.$$

This curve has cyclic symmetry $\mathtt{C}_{2n-2}$ from the appearance of ζ^{2n-2} and $\mathtt{C}_2$ due to the appearance of x^2. The genus of $\hat{\mathcal{C}}$ is $(2n-1)^2 - 2n$. The genus of $\tilde{\mathcal{C}}$ is $2n-1$. Finally $\tilde{\mathcal{C}}$ covers a genus $n-1$ curve $\mathcal{C}$

$$y^2 = P_n(u)^2 - 4\alpha^2u^2.$$

Here we expect the Toda motion to lie in the Prym of this covering, but the general theory warrants further study.

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Appendix A. Proof of Theorem 6.2 via Poincaré Reducibility

It is instructive to see an alternative proof of Theorem 6.2 in terms of Poincaré's reducibility condition, which we now recall. Consider Riemann matrices

$$\hat{\Pi} = \begin{pmatrix} \hat{\mathcal{A}} \\ \hat{\mathcal{B}} \end{pmatrix} = \begin{pmatrix} 1 \\ \hat{\tau} \end{pmatrix} \hat{\mathcal{A}}, \qquad \Pi = \begin{pmatrix} \mathcal{A} \\ \mathcal{B} \end{pmatrix} = \begin{pmatrix} 1 \\ \tau \end{pmatrix} \mathcal{A},$$

where $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ are the $\hat{g} \times \hat{g}$ matrices of $\hat{\mathfrak{a}}$-periods and $\hat{\mathfrak{b}}$-periods respectively for the curve $\hat{\mathcal{C}}$ with similarly named quantities for the curve $\mathcal{C}$. If $\{\hat{\gamma}_a\}_{a=1}^{2\hat{g}}$ is a basis for $H_1(\hat{\mathcal{C}}, \mathbb{Z})$, $\{\hat{\omega}_\mu\}_{\mu=1}^{\hat{g}}$ a basis of holomorphic differentials of $\hat{\mathcal{C}}$, and $\{\gamma_i\}_{i=1}^{2g}$ a basis for $H_1(\mathcal{C}, \mathbb{Z})$, $\{\omega_\alpha\}_{\alpha=1}^{g}$ a basis of holomorphic differentials of $\mathcal{C}$, these are related by

$$\pi_*(\hat{\gamma}_a) = M_a{}^i \, \gamma_i, \qquad \pi^*(\omega_\mu) = \hat{\omega}_\alpha \, \lambda^\alpha{}_\mu.$$

Here λ is complex $\hat{g} \times g$-matrix of maximal rank and M is a $2\hat{g} \times 2g$-matrix of integers of maximal rank. Then from

$$(M\Pi)_{a\mu} = M_a{}^i \oint_{\gamma_i} \omega_\mu = \oint_{\pi_*\hat{\gamma}_a} \omega_\mu = \oint_{\hat{\gamma}_a} \pi^*\omega_\mu = \oint_{\hat{\gamma}_a} \hat{\omega}_\alpha \, \lambda^\alpha{}_\mu = (\hat{\Pi}\lambda)_{a\mu}$$

we obtain Poincaré's reducibility condition

$$\hat{\Pi}\lambda = M\Pi. \tag{A.1}$$

For the cyclic homology basis and corresponding $\hat{\mathfrak{a}}$-normalized differentials $\hat{v}_i$ of Fay this takes the form

$$\begin{pmatrix} 1 \\ \hat{\tau}^c \end{pmatrix} \mathcal{I}' = \begin{pmatrix} \mathcal{I}' & 0 \\ 0 & \mathcal{I} \end{pmatrix} \begin{pmatrix} 1 \\ \tau^c \end{pmatrix} := M \begin{pmatrix} 1 \\ \tau^c \end{pmatrix},$$

where we define the $\hat{g} \times g$ matrices $\mathcal{I}$, $\mathcal{I}'$ and (to be used shortly) P,

$$\mathcal{I} = \begin{pmatrix} n & 0 \\ 0 & 1_{g-1} \\ \vdots & \vdots \\ 0 & 1_{g-1} \end{pmatrix}, \quad \mathcal{I}' = \begin{pmatrix} 1 & 0 \\ 0 & 1_{g-1} \\ \vdots & \vdots \\ 0 & 1_{g-1} \end{pmatrix}, \quad P = \begin{pmatrix} 1_g \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

For the same cyclic homology basis but an arbitrary basis of holomorphic differentials we obtain (A.1) with

$$\lambda = \hat{\mathcal{A}}^{-1}\mathcal{I}'\mathcal{A}.$$

Now bringing together the Ercolani-Sinha constraint (4.6) with (A.1) we find

$$-2(0, \ldots, 0, 1)\lambda = (\mathbf{n}, \mathbf{m}) \begin{pmatrix} \hat{\mathcal{A}} \\ \hat{\mathcal{B}} \end{pmatrix} \lambda = (\mathbf{n}, \mathbf{m}) M \begin{pmatrix} \mathcal{A} \\ \mathcal{B} \end{pmatrix} = (r_0, n\mathbf{r}, ns_0, n\mathbf{s}) \begin{pmatrix} \mathcal{A} \\ \mathcal{B} \end{pmatrix}.$$

where we have used (6.4) and that $(\mathbf{n},\mathbf{m})M=(r_0,n\mathbf{r},ns_0,n\mathbf{s})$. Here $\hat{\mathcal{A}}$ has been constructed from the differentials $\hat\omega_{r,s}=\zeta^r\eta^s d\zeta/\partial_\eta P(\eta,\zeta)$ (which are not Fay's normalized differentials $\hat v_i$) while the differentials for $\mathcal{A}$ are as yet unspecified and we wish to construct λ. Using (4.8) it is convenient to choose $u_s=-x^s dx/(ny)$ (so that $\pi^*(u_s)=\hat\omega_{n-2-s,s}$) and to order the differentials with the noninvariant differentials before the invariant differentials, $\{\hat\omega_{r,s}\}_{r+s\neq n-2}\cup\{\hat\omega_{n-2,0},\ldots,\hat\omega_{0,n-2}\}$. Then we find the matrix of periods

$$\hat{\mathcal{A}}=\begin{pmatrix} 0 & \ldots & 0 & * & \ldots & * \\ & \mathcal{D}^{(0)} & & & \mathcal{A}' & \\ & \mathcal{D}^{(1)} & & & \mathcal{A}' & \\ & \vdots & & & \vdots & \\ & \mathcal{D}^{(n-1)} & & & \mathcal{A}' & \end{pmatrix}.$$

Here the first row has zero entries for the periods of the noninvariant differentials over the invariant cycle $\hat{\mathfrak{a}}_0$ while $\mathcal{D}^{(k)}$ is the $(g-1)\times(\hat g-g)$ matrix of periods of the noninvariant differentials over the cycles $\hat{\mathfrak{a}}_{i+k(g-1)}$ ($i=1,\ldots,g-1$). Thus

$$\mathcal{D}^{(k)}_{i,(r,s)}=\int_{\mathfrak{a}_{i+k(g-1)}}\hat\omega_{r,s}=\int_{\phi^k(\mathfrak{a}_i)}\hat\omega_{r,s}=\int_{\mathfrak{a}_i}(\phi^k)^*\hat\omega_{r,s}=\omega^{k(r+s+2)}\int_{\mathfrak{a}_i}\hat\omega_{r,s}=\omega^{k(r+s+2)}\mathcal{D}^{(0)}_{i,(r,s)}.$$

The matrix of periods $\mathcal{A}'$ of the invariant differentials over the same cycles is such that

$$\int_{\hat{\mathfrak{a}}_i}\hat\omega_{n-2-s,s}=\int_{\hat{\mathfrak{a}}_i}\pi^*(u_s)=\int_{\pi_*(\hat{\mathfrak{a}}_i)}u_s=\int_{\mathfrak{a}_i}u_s,$$

and the matrix of periods $\mathcal{A}$ for the curve $\mathcal{C}$ appearing above is precisely the submatrix

$$\mathcal{A}=\begin{pmatrix} * & \ldots & * \\ & \mathcal{A}' & \end{pmatrix}.$$

Next we note that we may write

$$\begin{aligned}(0,\ldots,0,1)\lambda\mathcal{A}^{-1}&=(0,\ldots,0,1)\hat{\mathcal{A}}^{-1}\mathcal{I}'=(0,\ldots,0,1)\hat{\mathcal{A}}^{-1}CP=(0,\ldots,0,1)\left(C^{-1}\hat{\mathcal{A}}\right)^{-1}P\\ &=\left(\left(C^{-1}\hat{\mathcal{A}}\right)^{-1}_{\hat g,1},\ldots,\left(C^{-1}\hat{\mathcal{A}}\right)^{-1}_{\hat g,g}\right)\end{aligned}$$

with

$$C=\begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1_{g-1} & 0 & \ldots & 0 \\ 0 & 1_{g-1} & 1_{g-1} & \ldots & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 1_{g-1} & 0 & \ldots & 1_{g-1} \end{pmatrix}, \qquad C^{-1}=\begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1_{g-1} & 0 & \ldots & 0 \\ 0 & -1_{g-1} & 1_{g-1} & \ldots & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & -1_{g-1} & 0 & \ldots & 1_{g-1} \end{pmatrix}.$$

This factorization was motivated by the observation that

$$C^{-1}\hat{\mathcal{A}}=\begin{pmatrix} 0 & \ldots & 0 & * & \ldots & * \\ & \mathcal{D}^{(0)} & & & \mathcal{A}' & \\ & \mathcal{D}^{(1)}-\mathcal{D}^{(0)} & & & 0 & \\ & \vdots & & & \vdots & \\ & \mathcal{D}^{(n-1)}-\mathcal{D}^{(n-2)} & & & 0 & \end{pmatrix}=\begin{pmatrix}\mathcal{E} & \mathcal{A}\\ \mathcal{F} & 0\end{pmatrix}$$

and so upon noting that $\hat g-g$ is even and $|C^{-1}\hat{\mathcal{A}}|=|\mathcal{A}|\,|\mathcal{F}|$ we have the cofactor expression

$$\left(C^{-1}\hat{\mathcal{A}}\right)^{-1}_{\hat g,j}=\frac{1}{|\mathcal{A}|\,|\mathcal{F}|}\operatorname{Cof}\left(C^{-1}\hat{\mathcal{A}}\right)_{j,\hat g}=\frac{1}{|\mathcal{A}|}\operatorname{Cof}(\mathcal{A})_{j,g}=\mathcal{A}^{-1}_{g,j}.$$

Thus
$$(0,\ldots,0,1)\lambda\mathcal{A}^{-1} = (0,\ldots,0,1)\mathcal{A}^{-1}$$
where the right-hand row vector is g-dimensional and the left is $\hat{g}$-dimensional. Bringing these results together establishes the theorem.

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